Space-time correlations within pairs produced during inflation, a wave-packet analysis.

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Abstract

In homogeneous universes the propagation of quantum fields gives rise to pair creation of quanta with opposite momenta. When computing expectation values of local operators, the correlations between these quanta are averaged out and no space-time structure is obtained. In this article, by an appropriate use of wave packets, we reveal the space-time structure of these correlations. We show that every pair emerges from vacuum configurations which are torn apart so as to give rise to two semi-classical currents: that carried by the particle and that of its 'partner'. During inflation the partner's current lives behind the Hubble horizon centered around the particle. Hence any measurement performed within a Hubble patch would correspond to an uncorrelated density matrix, as for Hawking radiation. However, when inflation stops, the Hubble radius grows and eventually encompasses the partner. When this is realized the coherence is recovered within a patch. In this paper, we focus on the case of a massive field with rare pair creation events. However, our analysis also applies to cases leading to arbitrary high occupation numbers. Hence it could be applied to primordial gravitational waves and primordial density fluctuations.

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I. INTRODUCTION

In inflationary models, the large scale structure of the universe is of quantum origin: it results from pair creation processes induced by the expansion rate [1, 2]. These primordial fluctuations are at the same time very coherent and incoherent, depending on which properties one is looking at. On one hand, since the background space is homogeneous and since the initial state is vacuum, quanta characterized by different conformal momenta $\mathbf{k}$ are incoherent. On the other hand, since these quanta are created by pairs, there exist (EPR) correlations amongst the two partners in each pair. In fact, at the end of inflation, the two-mode states made of $\mathbf{k}$ and $-\mathbf{k}$ are highly squeezed, so squeezed that one generally abandons the quantum settings and works with a classical description of primordial fluctuations. Nevertheless, this classical description (in terms of random fluctuations) still incorporates the correlations between the $\mathbf{k}$ and the $-\mathbf{k}$ modes [3, 4].

In this article, we analyze an aspect which so far has received little attention, that of the space-time distribution of these correlations. This problem is not specific to primordial density fluctuations, it also applies to primordial gravitational waves and more generally to all pair creation processes in cosmology. To extract the space-time structure encoded in the two-modes states it is necessary to introduce wave packets. Indeed, since the background is homogeneous, expectation values of local operators are translation invariant and exhibit no specific space-time structure.

The first question to address thus concerns the use of wave-packets for studying pair creation. This question has been already addressed to study two related phenomena. It was introduced in [5, 6] to reveal the correlations between charged quanta produced in a constant electric field and was then applied to Hawking quanta in the context of black hole evaporation [6, 7]. In both cases, the space-time structure of the current carried by a single pair was obtained. The new ingredient which allows to consider this current is the projector Π which filters out the final configurations so as to isolate those associated with the pair under consideration. Physically, this filtering procedure can be conceived as resulting from a local measurement performed at late time (in the case of electro-production, it could correspond to detect an electron, or some electric current). One then compute $\langle \Pi J(t, \mathbf{x}) \rangle$ of some local operator J, typically a current. This expectation value gives the value of J which is conditional to find the final state associated with Π. In other words, $\langle \Pi J(t, \mathbf{x}) \rangle$ determines the space-time distribution of the configurations which are correlated to this final state. This distribution shows that to every produced particle (described by a local wave-packet) corresponds a well-localized partner which lives on the other side of the horizon. In the black hole case, this horizon coincides with the event horizon, whereas in the electric case, it is defined by the uniformly accelerated trajectory followed by the particle once it is on-shell.

When applying this procedure to cosmology, we find a similar picture. We shall proceed in two steps. We first consider the simple case in which pair creation events are rare. This is realized by considering a field whose mass m is much larger than the Hubble parameter H. In this case the mean occupation number is exponentially suppressed, of the order of $e^{-2\pi m/H}$. Secondly, we consider the opposite regime wherein the occupation number is very large. This is similar to primordial gravitational waves and fluctuations of the inflaton field. We shall see that the parameters which fix the space-time distribution of the correlations are *not* related to the mean occupation number. (The latter is determined

by the norm of the coefficients of the Bogoliubov transformation whereas the space-time distribution of the correlations are fixed by the phase of these coefficients.) Hence our analysis applies to arbitrarily high occupation number. In fact it equally applies to two-modes states which are described by a classical probability distribution.

We find that every produced pair of particles results from a dipole of vacuum configurations which is torn apart by the expansion of the universe and which gives rise to two on-shell currents. The creation occurs when the decreasing (physical) momentum is of the same order as the mass of the field. The temporal and spatial extension of this region is of the order of the Hubble radius H^{-1}, independently of the mass of the particle. (This is reminiscent to pair production in a constant electric field wherein the creation region [6] is of the order of the acceleration^{-1}.) We find that the current carried by the partner of every particle lies always on the other side of the Hubble radius centered around that particle. Hence any measurement performed within a patch would appear to result from a density matrix, as for Hawking radiation when one performs measurements outside the event horizon. However, when considering the evolution from a de Sitter period to an adiabatic one, the partners will progressively enter the Hubble radius. When they have entered the enlarged Hubble radius, local measurements will be sensitive to quantum correlations and interference patterns can be obtained.

In the regime of large mass, the role of the projector Π which isolates a given pair and the physical interpretation of the corresponding expectation value $\langle \Pi J(t, \mathbf{x}) \rangle$ are identical to those of electro-production: when a heavy particle is detected in a given space-time region, one can evaluate the value of the current (or the stress-energy) carried by this particle and its partner. In the other regime, when facing the production of a macroscopic number of light (or massless) quanta, the procedure should be slightly modified. The appropriate way to filter out the final configurations is now achieved by a projector Π which characterizes a local energy density fluctuation and no longer the particle content of the state. Having defined this projector, $\langle \Pi J(t, \mathbf{x}) \rangle$ tells us the location of the 'partner' energy fluctuation. We shall see that the macroscopic character of the occupation number does not erase the local character of these correlations. We can thus envisage to apply our analysis to physical cosmology wherein primordial gravitational waves and density fluctuations are described by highly squeezed two-modes states. This program is only sketched at the end of this paper and shall be further developed in a forthcoming work.

In section 2, we recall the basic steps involved in analyzing pair creation in cosmology. We pay special attention to identify the phases which are responsible for the space-time structure we are seeking. Then we proceed by an exact treatment of the modes and we conclude by an adiabatic treatment which allows to interpret the various results in semi-classical terms. In section 3, we show how to introduce wave-packets. We first present the results based on the exact solutions by several figures. We then return to the adiabatic treatment to explain the origin of their properties. In section 4, we briefly comment on the relations between our analysis and the correlations which lead to acoustic peaks in the CMB.

II. PAIR CREATION IN COSMOLOGY

A. General formalism and physical parameters

In this section we provide our notations and discuss the physical meaning of the three parameters which govern Bogoliubov coefficients in cosmology.

Throughout the paper, we consider field propagation in flat Roberson-Walker space-times. Their line element is given by

$$ds^2 \;=\; -dt^2 + a(t)^2 d\mathbf{x}^2\,, \tag{1}$$

with $\mathbf{x}$ the comoving position, t the cosmological time, and a the scale factor. The Hubble parameter is, as usual, given by $H = \partial_t a/a$. Let Φ be a massive complex scalar field minimally coupled to gravity. In terms of the proper time, the Klein-Gordon equation reads

$$\partial_t^2\Phi + 3H\partial_t\Phi - \frac{1}{a^2}\nabla^2\Phi + m^2\Phi = 0\,. \tag{2}$$

To eliminate the first order derivative term multiplied by H, we work with the rescaled field $\phi = a^{3/2}\Phi$. In addition, we decompose it into modes of given conformal wave-vector $\mathbf{k}$:

$$\phi(t,\mathbf{x}) = \int d^3k \frac{e^{i\mathbf{kx}}}{(2\pi)^{3/2}}\,\phi_{\mathbf{k}}(t)\,. \tag{3}$$

Then Eq. (2) reduces to a set of decoupled equations

$$(\partial_t^2 + \Omega_k^2)\phi_{\mathbf{k}}(t) = 0\,, \tag{4}$$

where $k = \sqrt{\mathbf{kk}}$ and where the time-dependent frequency is given by

$$\Omega_k^2(t) = \omega_k^2(t) - (\frac{3}{4}(\frac{\dot a}{a})^2 + \frac{3}{2}\frac{\ddot a}{a}),\ \text{ with } \omega_k^2(t) = m^2 + \frac{k^2}{a^2}\,. \tag{5}$$

In second quantization, the Fourier components $\hat\phi_{\mathbf{k}}$ of the rescaled field operator are decomposed as

$$\hat\phi_{\mathbf{k}}(t) = \phi_k(t)\,\hat a_{\mathbf{k}} + \phi_k^*(t)\,\hat b_{-\mathbf{k}}^\dagger\,, \tag{6}$$

where $a_{\mathbf{k}}$ and $b_{-\mathbf{k}}^\dagger$ are respectively the annihilation operator of a particle and the creation operator of the corresponding antiparticle with opposite momentum. Moreover, since the situation is isotropic, we can and shall work with time dependent modes $\phi_k(t)$ which depend only on the norm of $\mathbf{k}$. To define the creation and destruction operators, one needs to choose positive and negative norm solutions of Eq. (2). In terms of the rescaled field, the Klein-Gordon scalar product reads

$$\begin{aligned}\langle\phi_{k'}\frac{e^{i\mathbf{k}'\mathbf{x}}}{(2\pi)^{3/2}}, \phi_k\frac{e^{i\mathbf{kx}}}{(2\pi)^{3/2}}\rangle \;&=\; \int \frac{d^3x}{(2\pi)^3}\; e^{i(\mathbf{k}-\mathbf{k}')\mathbf{x}}\,\phi_{k'}^* i\overleftrightarrow{\partial_t}\,\phi_k \\ &=\; \delta^3(\mathbf{k}-\mathbf{k}')\;(\phi_k,\phi_k)\,, \end{aligned}\tag{7}$$

where (,) is the Wronskian. In conformity with the particle interpretation, we work with modes ϕ_k of unit Wronskians: $(\phi_k, \phi_k) = 1$ and $(\phi_k^*, \phi_k^*) = -1$. The (time independent) creation and destruction operators are defined by

$$\hat{a}_{\mathbf{k}} = \left(\phi_k, \hat{\phi}_{\mathbf{k}}\right), \qquad \hat{b}^\dagger_{-\mathbf{k}} = -\left(\phi_k^*, \hat{\phi}_{\mathbf{k}}\right) . \tag{8}$$

In virtue of the equal-time commutation relations

$$[\hat{\phi}_{\mathbf{k}}(t), \partial_t \hat{\phi}^\dagger_{-\mathbf{k}'}(t)] = i\delta^3(\mathbf{k} - \mathbf{k}') , \tag{9}$$

they obey

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = [b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}') . \tag{10}$$

Using these operators, the $\mathbf{k}$-th component of the vacuum is defined by the 'two-modes' ground state

$$|0_{\mathbf{k}}\rangle = |0_{\mathbf{k}}, p\rangle \otimes |0_{-\mathbf{k}}, a\rangle , \tag{11a}$$

$$a_{\mathbf{k}}|0_{\mathbf{k}}, p\rangle = 0, \quad b_{-\mathbf{k}}|0_{-\mathbf{k}}, a\rangle = 0 . \tag{11b}$$

In a static universe, this would be the end of the story. However since the frequency Ω_k is time dependent, the notion of positive and negative frequency modes is ambiguous. Hence, to set in vacuum at early time T_{in} and to read out the particle content at late time T_{out}, we must introduce two sets of positive frequency modes of Eq. (2): in modes defined at T_{in} and out modes at T_{out}. Because of the homogeneity of the background space, the Bogoliubov transformation which relates them reduces to a set of coefficients which depend on k only:

$$\phi_k^{out}(t) = \alpha_k \; \phi_k^{in}(t) + \beta_k^* \; \phi_k^{in*}(t) . \tag{12}$$

By definition, α_k and β_k are given by the (conserved) Wronskians

$$\alpha_k = \left(\phi_k^{in}, \phi_k^{out}\right), \quad \beta_k^* = -\left(\phi_k^{in*}, \phi_k^{out}\right) . \tag{13}$$

Then, as in Eq. (8), in and out operators are defined by in and out modes respectively. These two sets of operators are also related by a Bogoliubov transformation, and they define the in and out vacua, $|0, in\rangle, |0, out\rangle$ as the tensorial product over $\mathbf{k}$ of two-modes ground states $|0_{\mathbf{k}}, in\rangle$ and $|0_{\mathbf{k}}, out\rangle$ defined as in Eq. (11).

The normalization of the current $(\phi_k, \phi_k) = 1$ for both in and out modes implies that

$$|\alpha_k|^2 - |\beta_k|^2 = 1 . \tag{14}$$

This relation reduces to three the number of independent parameters. A convenient way to parameterize α_k and β_k is provided by

$$\alpha_k = e^{i\theta_k} \; \mathrm{ch} r_k , \tag{15a}$$

$$\beta_k = e^{i(\theta_k + 2\psi_k)} \; \mathrm{sh} r_k . \tag{15b}$$

The relationship with the conventional way to describe squeezed states is provided in Appendix B. To physically interpret the angles θ_k and ψ_k requires some care as they do

not possess invariant meaning. Instead, r_k is easily identified since the mean occupation number of out quanta in the in vacuum is given by

$$\langle 0in|a_{\mathbf{k}}^{out\dagger}a_{\mathbf{k}'}^{out}|0in\rangle = \delta^3(\mathbf{k}-\mathbf{k}')\,|\beta_k|^2 = \delta^3(\mathbf{k}-\mathbf{k}')\,\mathrm{sh}^2 r_k\,. \tag{16}$$

A similar interpretation of θ_k and ψ_k cannot be reached. Indeed, by an appropriate choice of the arbitrary phases of in and out modes, both θ_k and ψ_k can always be gauged away to zero. In spite of this, as we shall see in the next sections, θ_k and ψ_k contain physical information, *given* the two sets of in and out modes. To understand this, it is appropriate to work with modes such that the phase of ϕ_k^{in} is zero at T_{in} and that ϕ_k^{out} at T_{out}. In these settings, the roles of θ_k and ψ_k are easily identified.

The phase of α_k governs the evolution of the modes from T_{in} to T_{out}. Indeed, in the adiabatic limit, see Section 2.3 for details, one obtains

$$\theta_k = \int_{T_{in}}^{T_{out}} dt\ \omega_k(t)\,, \tag{17}$$

which is the classical action from T_{in} to T_{out}. Thus when considering as initial state a superposition of in states with different occupation numbers, and when re-expressing this superposition in terms of out states, the changes of the *relative* phases between the various components of the state are governed by θ_k. From this fact we learn that when the (Heisenberg) state is the in vacuum, θ_k is no longer accessible. This is why it drops out in inflationary models when the state of the primordial fluctuation modes is taken to be the (Bunch-Davies) vacuum [2, 8]. However when the initial state contains some superposition of excited states [9–11], or two inflationary phases [12], the power spectrum possesses rapid oscillations which are governed by θ_k.

The second phase ψ_k contains the information which specifies when the $\mathbf{k},-\mathbf{k}$ pair is created. To show this is one of the main purpose of Section 2.3. One can already notice that ψ_k is fixed in inflationary scenarios. However this is some how hidden by the (simple and appropriate) convention which consists in putting to zero the decaying mode, i.e. of keeping only the constant mode before re-entry, when the wavelength of the mode is still larger than the Hubble radius [13].

In the following, we will be needing the expression of the in vacuum as a superposition of out states:

$$|0_{\mathbf{k}},\ in\rangle = \frac{1}{|\alpha_k|}\exp\left(-\frac{\beta_k}{\alpha_k}a_{\mathbf{k}}^{out\dagger}b_{-\mathbf{k}}^{out\dagger}\right)|0_{\mathbf{k}},\ out\rangle\,. \tag{18}$$

The derivation of this equation can be found in Appendix B. From Eq. (18) we see that *only* ψ_k appears in the relative phases between out states with different occupation numbers.

B. The exact modes

In a flat de Sitter space, the conformal factor obeys $a(t) = \frac{1}{H}e^{Ht}$. In this case, the exact solutions of Eq. (4) are given by Bessel functions. The in-modes of unit positive Wronskian are given by

$$\phi_k^{in}(t) = \frac{\sqrt{\pi}}{2\sqrt{H}}e^{-\nu\pi/2}\mathcal{H}_{i\nu}^{(1)}(ke^{-Ht})\ , \tag{19}$$

where $\nu = \sqrt{m^2/H^2 - 9/4}$ and where $\mathcal{H}^{(1)}_{i\nu}$ is a Hankel function of the first kind. The identification of this solution with an in mode follows from the early time development of $\mathcal{H}^{(1)}_{i\nu}(z)$. For $|z| \to \infty$ at fixed ν [14] one has

$$\mathcal{H}^{(1)}_{i\nu}(z) \to \sqrt{\frac{2}{\pi z}} e^{i(z-\frac{\pi}{4})}\ e^{\nu\pi/2} . \tag{20}$$

In terms of the conformal time $\eta = -e^{-Ht}$, one verifies that ϕ^{in}_k describes excitations of the Bunch-Davis vacuum [8] as it obeys $\phi^{in}_k/\sqrt{a} \to e^{-i|k|\eta}/\sqrt{2k}$.

The out-modes of unit positive Wronskian are given by

$$\phi^{out}_k(t) = \frac{\sqrt{\pi/H}}{\sqrt{2\sinh(\pi\nu)}} J_{i\nu}(ke^{-Ht}) . \tag{21}$$

The identification follows from the asymptotic late time behavior of $J_{i\nu}$. When $|z| \to 0$, one gets

$$J_{i\nu}(z) \to \frac{\left(\frac{1}{2}z\right)^{i\nu}}{\Gamma(1+i\nu)} . \tag{22}$$

Hence, in the late future and up to a constant phase, ϕ^{out}_k tends to $e^{-imt}/\sqrt{2m}$ which corresponds to a comoving massive particle. More details about the asymptotic limits and the notion of particule can be found in subsection C. Using $|\Gamma(1+i\nu)|^2 = \pi\nu/\sinh\pi\nu$, one verifies that ϕ^{out}_k has a unit Wronskian for $\nu > 0$. In this paper we shall restrict ourselves to $\nu > 0$. Hence we shall not analyze minimally coupled massless fields which correspond to $i\nu = 3/2$. We shall return to this interesting case in a forthcoming work.

Having identify the modes, the Bogoliubov coefficients defined in Eq. (13) are immediately given by the relation

$$e^{-\nu\pi/2}\mathcal{H}^{(1)}_{i\nu}(z) = \frac{e^{\nu\pi/2}J_{i\nu}(z) - e^{-\nu\pi/2}J_{-i\nu}(z)}{\sinh(\nu\pi)} . \tag{23}$$

One gets real k-independent coefficients given by

$$\alpha = \frac{1}{\sqrt{1-e^{-2\pi\nu}}} , \quad \beta = \alpha e^{-\pi\nu} . \tag{24}$$

The k-independence expresses the stationarity of the process and stems from the fact that H is constant. Hence a change in k can be absorbed by a harmless shift in time. It should also be pointed out that we are working in a 'gauge' wherein both θ_k and ψ_k are zero.

Before examining the adiabatic limit, we briefly discuss the space-time behaviour of these modes. As such both in and out modes are completely delocalized. However one easily obtains the space time behaviour they encode by forming wave-packets:

$$\bar{\phi}_{\bar{k},\bar{x}}(t,x) = \int_{-\infty}^{\infty} dk\ f(k;\bar{k})\ \phi_k(t)\ e^{ik(x-\bar{x})} . \tag{25}$$

We work for simplicity in $1+1$ dimension and we designate by $\bar{k}$ the mean conformal momentum. Given the phase conventions we have adopted, $\bar{x}$ corresponds to the position

of the 'center of mass' of the pair of quanta involved in the waves packets. This will become clear in the sequel. In order to get simple and analytic expressions for $\bar{\phi}$, we choose $f(k;\bar{k}) = \theta(k)e^{-k/\bar{k}}$. This enables us to use Eq. 6.621-4 of [15]

$$\int_0^\infty dk\; e^{-ck} J_{i\nu}(bk) = b^{-i\nu} \left(\frac{\sqrt{c^2+b^2}-c}{\sqrt{c^2+b^2}} \right)^{i\nu} . \tag{26}$$

In Figures 1 and 2, we present the current $\bar{J} = i\bar{\phi}^* \overleftrightarrow{\partial_t}\, \bar{\phi}$ carried by a wave-packet built with out modes in the adiabatic regime, $|\beta| \ll 1$, and for the large occupation number $|\beta| \gg 1$ respectively. The corresponding wave-packets built with in modes exhibit less clear features because, for large t, the conformal distance between the particle and its partner becomes comparable to the spread of the wave-packet, see Appendix A. For this reason, we have plotted the norm of $\bar{\phi}$ (Fig. 3) in order to show the interferences between the two outgoing components.

In both regimes, at early times, one clearly sees the classical trajectory of the particle and that of its partner. These two trajectories are symmetrically distributed with respect to $\bar{x}$, put to zero in all figures. It should be stressed that wave packets of out modes (or in modes) characterize both the semi-classical trajectory of the particle and that of its partner. Indeed this pairing in no way depends on the choice of the function $f(k,\bar{k})$ but results from the creation process itself. It should also be stressed that, as such, the current $\bar{J}$ or the norm $|\bar{\phi}|^2$ have no physical meaning. They do not emerge from expectation values of operators and should thus be conceived as providing only a pictorial understanding of the local character of wave packets. On the contrary, the (more complicate) expressions we shall use in Section 3 do follow from expectation values of operators and possess a well defined physical meaning.

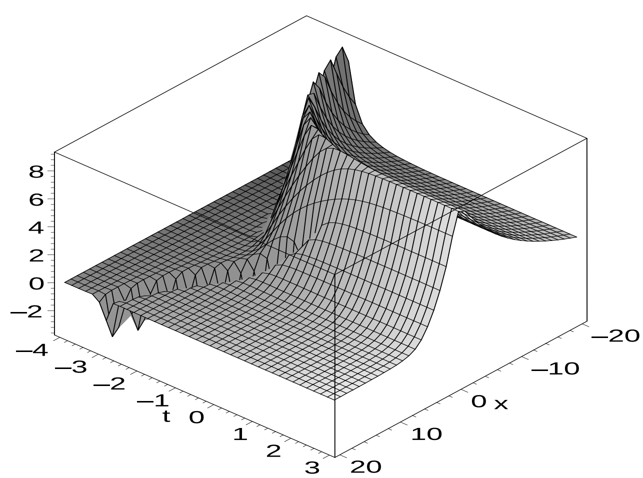

FIG. 1: The current carried by a wave-packet built with out modes in the adiabatic regime, for $|\beta|^2 = e^{-2\pi}$, $\nu = 1$, as a function of the conformal coordinate x and the cosmological time t. One clearly sees the incoming particle coming from negative x and its partner coming symmetrically from positive x. Both follow classical trajectories characterized by their mean momenta $\bar{k} = \pm 1$ respectively; see App. A. The ratio of the incoming currents is given by $-|\alpha/\beta|^2$. For large positive t one just has the unit current carried by the outgoing particle.

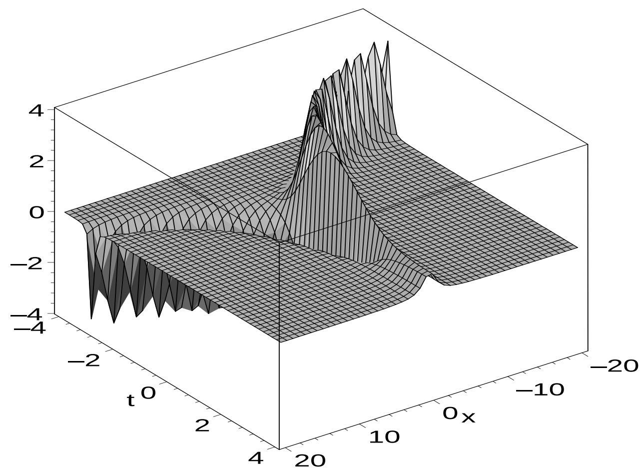

FIG. 2: The out current as a function of x and t, for $\bar{k} = 1, \bar{x} = 0$ and $\nu = 0.01$, i.e. very far from the adiabatic regime. Nevertheless the semi-classical properties of wave packets are robust. The incoming currents of respective values $|\alpha|^2$ and $-|\beta|^2$ are now of the same order. For large positive t one still has the unit current of the particle.

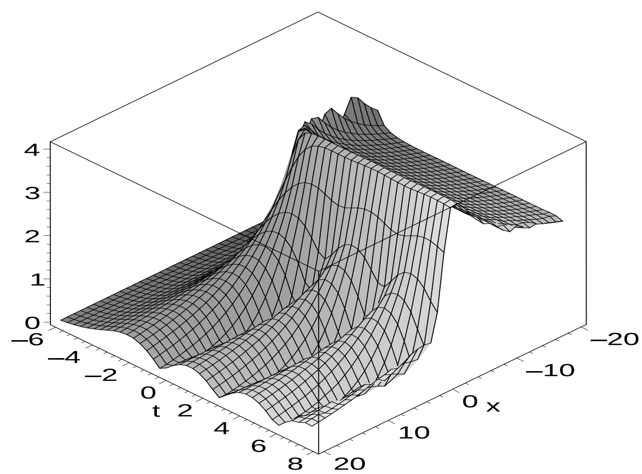

FIG. 3: The norm of a wave-packet mode built with in modes for $\nu = 1, \bar{k} = 1$. In addition to the incoming particle which enters from negative values of x, there are oscillations for positive x and from $t \simeq 0$, when the particle stops to be relativistic. These oscillations arise from the interferences between the two out wave packets in the norm $|\alpha\bar{\phi}_{out} + \beta\bar{\phi}^*_{out}|$. (They have been amplified by using a non-linear vertical scale.) Deeper in the adiabatic regime, $\beta \simeq e^{-\pi\nu} \ll 1$, they are exponentially suppressed, whereas in the opposite limit $\beta/\alpha \to 1$ the picture becomes more symmetrical in $x, -x$ since both terms in the norm become of the same order.

C. The adiabatic limit

The usefulness of considering the adiabatic regime is that it provides a neat semi-classical description of the creation process. We shall see indeed that the value of the Bogoliubov coefficient β is given, as in a tunneling effect, in terms of the exponential of the Hamilton-Jacobi action evaluated along a well-defined trajectory. Moreover we shall learn from this expression when the creation of the pair occurs, thereby explaining the

properties of the former figures. From this identification, we shall also be able to identify the modes which are subject to pair creation (and those which are not) when the inflation lasts a finite time.

The adiabatic regime is obtained when the mass of the field m is much larger that the Hubble constant H. In this limit, the frequency Ω_k of Eq. (4) can be replaced by the 'classical' frequency $\omega_k = \sqrt{k^2/a^2 + m^2}$ since the differences scale as $(H/m)^2$. In this regime, the WKB approximation is good. Indeed, its validity is controlled by the dimensionless function

$$\lambda_k(t) = \frac{\partial_t \omega_k}{\omega_k^2} = -\frac{H}{m}\left(\frac{ma}{k}\frac{1}{(1 + m^2a^2/k^2)^{\frac{3}{2}}}\right) . \tag{27}$$

The maximum of $|\lambda|$ is given by $(H/m)\sqrt{12}/9$. It is reached when $k/a = m\sqrt{2}$, i.e., when the physical momentum is of the same order as the mass of the field. These two results are exact when H is a constant but are also valid provided $\dot{H}/H^2 \ll 1$. In the rest of this Section, we shall consider only the de Sitter case. The generalisation of the results, such as Eq. (45), is obtained by replacing H by $H(k) = H(t_{tp})$, where $t_{tp}(k)$ is the solution of $k/a(t) = m$.

It is interesting to notice that $\lambda_k \to 0$ in both asymptotic regimes ($t \to \pm\infty$) for different reasons: For $t \to -\infty$, the particle goes on the light cone and the mass contribution to its energy becomes insignificant. Hence it becomes asymptotically decoupled from the expansion because of conformal invariance. In the other regime instead, for $t \to \infty$, one obtains a heavy particle at rest with respect to the cosmological frame and asymptotically insensitive to the expansion rate.

In brief, as in the former subsection, we see that one reaches adiabaticity for asymptotic times, for all values of $\nu > 0$ to be precise. Hence the identification of asymptotic in and out modes is unambiguous. In the adiabatic regime, when $\nu \gg 1$, one has $\lambda_k(t) \ll 1$ for all times since $\lambda < \nu^{-1}$. From now on we work in this regime. Then the WKB mode $\tilde{\phi}_k(t)$

$$\tilde{\phi}_k(t) = \frac{1}{\sqrt{2\omega_k(t)}} e^{-i\int_{T_{in}}^t dt' \omega_k(t')} , \tag{28}$$

is a good approximation of positive frequency solutions of Eq. (4). This can be seen from

$$i\partial_t \tilde{\phi}_k(t) = \left(1 - i\frac{\lambda_k(t)}{2}\right)\omega_k(t)\tilde{\phi}_k(t) . \tag{29}$$

Hence it is appropriate to express exact solutions of Eq. (4) as linear superpositions of WKB modes:

$$\phi_k(t) = c_k(t)\tilde{\phi}_k(t) + d_k(t)\tilde{\phi}_k^*(t) . \tag{30}$$

Owing to the previous discussion, $\tilde{\phi}_k$ becomes[1] an exact solution for $t \to -\infty$. Hence, the

[1] One can evaluate the residual effect engendered by the fact that the limit $T_{in} \to -\infty$ has not been taken. One finds [11] that the adiabatic vacuum enforced at T_{in} differs from the Bunch-Davies vacuum by Bogoliubov coefficients $\beta_{T_{in},-\infty} \simeq \lambda_k(T_{in}) \simeq \nu^{-1}e^{-H|T_{in}-t_{tp}|}$ where $t_{tp}(k)$ is defined by Eq. (43). Similarly, the fact of imposing vacuum at some finite late time defines out modes which differ from the asymptotic ones by Bogoliubov coefficients $\beta_{T_{out},\infty} \simeq \lambda_k(T_{out}) \simeq \nu^{-1}e^{-2H(T_{out}-t_{tp})}$. Since the residual effects fall off exponentially fast both for early and late times, the notion of asymptotic quanta is well defined in the present case. Hence, pair creation amplitudes are also well defined.

exact positive frequency solution $\phi_k^{T_{in}}$ whose phase vanishes at T_{in} (large and negative) is given by Eq. (30) with

$$c_k(T_{in}) = 1, \quad d_k(T_{in}) = 0 \,. \tag{31}$$

Similarly, since $\tilde{\phi}_k$ is also an exact solution of Eq. (4) in the asymptotic future, it asymptotically coincides with the exact out mode $\phi_k^{T_{out}}(t)$, up to an arbitrary phase. We fix this phase by requiring that the phase of $\phi_k^{T_{out}}$ vanishes for $t = T_{out}$. Hence, for large positive t, one has

$$\phi_k^{T_{out}}(t) = \frac{1}{\sqrt{2\omega_k(t)}} e^{i\int_t^{T_{out}} dt' \omega_k(t')} = e^{i\int_{T_{in}}^{T_{out}} dt' \omega_k(t')}\, \tilde{\phi}_k(t)\,. \tag{32}$$

Moreover, since $c(t)$ and $d(t)$ become constant one can write

$$\phi_k^{T_{in}}(t) = c_k(T_{out})\ \tilde{\phi}_k(t) + d_k(T_{out})\ \tilde{\phi}_k^*(t)\,. \tag{33}$$

Together with Eq. (32), this equation gives us the Bogoliubov coefficients between $\phi_k^{T_{in}}$ and $\phi_k^{T_{out}}$. Using their definitions Eq. (13), we get

$$\alpha_k = c_k^*(T_{out})\ e^{i\int_{T_{in}}^{T_{out}} dt' \omega_k(t')}\,, \tag{34a}$$

$$-\beta_k = d_k^*(T_{out})\ e^{-i\int_{T_{in}}^{T_{out}} dt' \omega_k(t')}\,. \tag{34b}$$

To compute $c_k(T_{out})$ and $d_k(T_{out})$, one should determine how they evolve from T_{in} to T_{out}. Their evolution is completely fixed by requiring that, for all t, one has (see [16] for details)

$$i\partial_t \phi_k^{T_{in}} = \omega_k \left(c_k(t)\tilde{\phi}_k - d_k(t)\tilde{\phi}_k^* \right) \,. \tag{35}$$

Then the conservation of the Wronskian gives

$$(\phi_k^{T_{in}}, \phi_k^{T_{in}}) = |c_k(t)|^2 - |d_k(t)|^2 = 1\,, \tag{36}$$

and Eq. (4) leads to

$$\partial_t c_k = \frac{\partial_t \omega}{2\omega} e^{i2\int_{T_{in}}^t dt' \omega(t')}\, d_k(t)\,, \tag{37a}$$

$$\partial_t d_k = \frac{\partial_t \omega}{2\omega} e^{-i2\int_{T_{in}}^t dt' \omega(t')}\, c_k(t)\,. \tag{37b}$$

Up to now no approximation has been used in the search of the evolution of c and d. However the equations have been recast in a form which is most appropriate to evaluate perturbatively non-adiabatic effects. Indeed, in the zeroth order approximation, one neglects λ_k, hence $c_k = 1$ and $d_k = 0$ for all t. This is the usual WKB approximation. To first order in the non-adiabaticity, one puts $c_k = 1$ in Eq. (37) and one integrates it to get

$$d_k(T_{out}) = \int_{T_{in}}^{T_{out}} dt'\ \frac{\partial_{t'} \omega_k}{2\omega_k} e^{-2i\int_{T_{in}}^{t'} dt \omega_k(t)}\,. \tag{38}$$

Given that $\lambda_k \ll 1$, one can evaluate this integral by a (kind of) saddle point approximation. One search for the time $t_c(k)$ in the complex plane where $\omega_k(t_c) = 0$ and one evaluates the phase of the integrand at that time to get

$$d_k(T_{out}) = -ie^{-2i\int_{T_{in}}^{t_c} dt\ \omega_k(t)}\,. \tag{39}$$

The evaluation of the prefactor is rather delicate, see [17] for the details. This adiabatic result is valid if the domain $[T_{in}, T_{out}]$ is wide enough so that the saddle time $t_c(k)$ is well enclosed. If not, one gets boundary contributions which decrease like $\lambda_k(T_{in})$ and $\lambda_k(T_{in})$, see the former footnotes. Using Eqs. (15b, 34b) we can rewrite $d_k(T_{out})$ as

$$d_k(T_{out}) = -\beta_k^* e^{-i\theta_k} \simeq -r_k e^{-2i(\psi_k+\theta_k)} = -e^{-2i(\psi_k'+\theta_k)}\,, \tag{40}$$

where we have anticipated the fact $r_k \ll 1$ in the adiabatic regime. We have also introduced the complex phase $\psi_k' = \psi_k + (i/2)\ln r_k$.

In brief, to first order in the non-adiabaticity, and when working with the convention that the phase of $\phi^{T_{in}}$ and $\phi^{T_{out}}$ vanish at T_{in} and T_{out} respectively, θ_k and ψ_k' are given by Eq. (17) and

$$\psi_k' = -\int_{t_c(k)}^{T_{out}} dt\, \omega_k(t) - \frac{\pi}{4}\,. \tag{41}$$

This equation shows that there is a simple and universal expression for ψ_k' in terms of the Hamilton-Jacobi action evaluated along the semi-classical trajectory which originates from t_c where ω_k vanishes. The mean occupation number is fixed by the imaginary part of this action. More importantly for us, the real part of ψ_k' $(= \psi_k)$ fixes, as we shall see, the space-time properties of the correlations induced by pair creation.

In a flat de Sitter space, all the previous expressions can be evaluated. The location of the saddle time given by $a(t_c) = H^{-1}e^{Ht_c} = -ik/m$ furnishes [2]

$$t_c(k) = \frac{1}{H}\ln(\frac{Hk}{m}) - i\frac{\pi}{2H}\ . \tag{42}$$

For later convenience we introduce the 'turning-point'

$$t_{tp}(k) = \Re(t_c) = \frac{1}{H}\ln(\frac{Hk}{m})\ , \tag{43}$$

where $k/a(t_{tp}) = m$. From Eq. (43) we learn that the pair is preferably created when the adiabatic parameter λ_k is near its maximum value, when the particle stops to be relativistic. (A pictorial interpretation of the turning point can be reached by adopting the somewhat artificial coordinate $(t, x) \to (t, \sqrt{a}x)$. Then the classical trajectory (see Eq. (A7)) possesses a turning point at t_{tp}.)

Using Eq. (A4), we get

$$\psi_k' = -\left[F_k(T_{out}) - F_k(t_c)\right] - \frac{\pi}{4} = -F_k(T_{out}) - \frac{\pi}{4} - i\frac{\pi}{2}\frac{m}{H}\,. \tag{44}$$

[2] The similarity of this equation with the corresponding saddle point condition governing the Unruh effect and black hole evaporation is remarkable, see Eqs. (2.49) and (3.43) in [6]. In all cases, the choice of the imaginary sign of the saddle time is such that it leads to exponentially suppressed amplitudes.

The real part of ψ_k' is governed by $F_k(T_{out})$, the primitive of $\omega_k(t)$ evaluated at T_{out}. This results from the fact that the real part of $F_k(t)$ vanishes when evaluated at the creation time $t_c(k)$. On the other hand, the imaginary part of ψ_k' comes entirely from $F_k(t_c) = -i\pi m/2H$ which determines the occupation number. In fact one finds

$$|d_k(T_{out})|^2 = |\beta_k|^2 = e^{-2\pi m/H} \,, \tag{45}$$

in agreement with Eq. (24) in the limit $m/H \gg 1$.

As a last application of the semi-classical treatment, we show that the identification of the creation time $t_c(k)$ allows for an evaluation of the decay probability of the vacuum when considering the quantization in a space-time box of proper length L_P and of time interval $\Delta T = T_f - T_i$. The probability that no pair is created is

$$|\langle 0out|0in\rangle|^2 = \prod_k \frac{1}{|\alpha_k|^2} = \exp\left(-\sum_k \ln(1+|\beta_k|^2)\right) \,, \tag{46}$$

where one should sum only over relevant modes, that is, modes which contribute to the decay of the vacuum given the space-time box. When the size of the box is large enough, i.e. $L_P \gg 1/H$ and $\Delta T \gg 1/H$, one can neglect the edge effects and separate the modes according to whether or not their saddle time $t_c(k)$ falls within the interval $\Delta T = T_f - T_i$, see [6] for the same analysis applied to the Schwinger effect.

The modes with their turning point which falls outside the interval $[T_i, T_f]$ do not contribute since the creation process occurs around $t_{tp}(k)$. Instead, modes with their turning point that lies within that interval participate to the above sum. Their number is

$$\sum_k = 2\int_{k(T_i)}^{k(T_f)} \frac{dk}{2\pi}\frac{L_P}{a(t_{tp})} = 2\frac{mL_P}{2\pi}\int_{k(T_i)}^{k(T_f)} \frac{dk}{k} = 2\frac{mHL_P\Delta T}{2\pi} \,. \tag{47}$$

The factor of 2 arises from the fact that we are dealing with a complex field. The integrand of the first integral ($= L_P/2\pi a$) is the time dependent density of conformal modes in a box of fixed proper length. We then reexpress it as a function of k using $t_{tp}(k)$. The upper and lower values of the integral are given by $k(t)$, the inverse function of $t_{tp}(k)$, evaluated at T_f and T_i.

As one might have expected, the decay of the vacuum is governed by an extensive quantity, namely the number of 'relevant' modes in a box of space time volume $L_P\Delta T$. When inserting Eq. (46) in Eq. (47), one obtains the equivalent of the famous Schwinger formula [6]:

$$|\langle 0out|0in\rangle|^2 = \exp\left(-\frac{ELT}{2\pi}\ln(1+e^{-\pi m^2/E})\right) \,, \tag{48}$$

where $E = ma$ is the constant electric force acting on charged quanta. Notice however that the way Eq. (47) is obtained, i.e. through an appeal to exponentially growing wave vectors (corresponding to initial trans-Planckian physical momenta), is identical to that governing vacuum instability giving rise to black hole radiation [6].

III. SPACE-TIME CORRELATIONS

A. The absence of structure in expectation values

When working in the in vacuum, expectation values of local operators such as that of the current $\langle J_\mu(x)\rangle$ or the stress-energy tensor $\langle T_{\mu\nu}(x)\rangle$ do not depend on $\mathbf{x}$ because of space homogeneity. Two-point functions such as $\langle J_\mu(x)J_\nu(y)\rangle$ or $\left\langle T_{\mu\nu}(x)T_{\mu'\nu'}(y)\right\rangle$ are already more interesting as they correlate field configurations at different locations. Hence they are sensitive to the entanglement in the two modes states $\mathbf{k}, -\mathbf{k}$. However since all $\mathbf{k}$ participate with the same weight in these vacuum expectation values, very little structure is finally obtained.

It is a worth exercise to compute this residual structure. Indeed it brings out the physically relevant phases irrespectively of the adopted conventions. Moreover it provides a neat introduction to the forthcoming analysis of wave packets. To determine this residual structure it is sufficient to analyze the Green function as the above mentioned two-point functions can be derived from it. For further simplicity we work at equal time and focus on its spatial dependence. In the in vacuum Green function is given by

$$\begin{aligned}\langle 0_{in}|\hat{\phi}(t,\mathbf{x})\hat{\phi}(t,\mathbf{0})^\dagger|0_{in}\rangle &= \int\frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{kx}}\, |\phi_k^{in}(t)|^2 \\ &= \int\frac{d^3k}{(2\pi)^3} e^{i\mathbf{kx}}\left[(|\alpha_k|^2+|\beta_k|^2)|\phi_k^{out}(t)|^2 - 2\Re\left\{\alpha_k^*\beta_k(\phi_k^{out}(t))^2\right\}\right] \end{aligned} \quad (49)$$

Using the phases introduced in Eq. (15) and the fact that in de Sitter space the norm of Bogoliubov coefficients are k-independent, one has

$$\langle 0_{in}|\hat{\phi}(t,\mathbf{x})\hat{\phi}(t,\mathbf{0})^\dagger|0_{in}\rangle = |\alpha|^2\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{kx}}\left[(1+|\frac{\beta}{\alpha}|^2)|\phi_k^{out}|^2 - 2|\frac{\beta}{\alpha}|\Re\left\{(e^{i\psi_k}\phi_k^{out})^2\right\}\right] \quad (50)$$

Two interesting limits can be considered. That governing the adiabatic limit and the opposite case in which $\beta/\alpha \to 1$. We shall restrict ourselves to the first case in this section. In this case, the first term in Eq. (50) reduces to the flat result, i.e. the Yukawa potential. In 1+1 dimensions, for large mx_p, it is given by $e^{-mx_p}/\sqrt{mx_p}$ where $x_p = |a(t)x|$ is the proper distance at time t. In the adiabatic regime, using Eq. (40) , the second term is given by the real part of

$$-\int\frac{dk}{2\pi}\, e^{ikx}(e^{i\psi_k'}\phi_k^{out}(t))^2 \;=\; i\int\frac{dk}{2\pi}\,\frac{e^{ikx}}{2\omega_k(t)}e^{-i2\int_{t_c(k)}^t dt'\;\omega_k(t')}\,. \quad (51)$$

We see that the arbitrary phases associated with T_{in} and T_{out} drop out from the product $e^{i\psi_k'}\phi_k^{out}(t)$. This is as it should be since the phase of this product is measurable. It determines indeed the sign of the corrections to the vacuum term. To evaluate the integral, we use again a saddle-point approximation [18]. One thus searches for the solution of

$$\partial_k\left(kx - 2\int_{t_c(k)}^t dt'\;\omega_k(t')\right)|_{k=k^*} = 0\,, \quad (52)$$

in complex k-plane. One gets

$$x = 2x_k(t)|_{k=k^*}\,. \quad (53)$$

That is, twice the classical displacement introduced in Appendix A, see Eq. (A8). The saddle-point approximation is valid well outside the Hubble radius, i.e. $a(t)x \gg 1/H$. In this case, k^* is always real. It gives the value of k such that, when the particle is found at $x = 0$ at t, its partner is at x at the same time. Thus the function $2x_k(t)$ gives the (mean) distance between the particle and its partner when they are created from vacuum. This will be clarified in the next Section. When $xa(t) \gg 1/H$, one asymptotically gets $k^*x \sim 2m/H$. Then the second term of the Green function behaves as

$$e^{-\pi m/H} \frac{1}{\sqrt{mHx^2}} \cos\left(2mt + \frac{2m}{H}\ln(x)\right) . \tag{54}$$

Contrary to the Yukawa term, the amplitude of this term decreases only as the inverse of the conformal distance x. (This slow decrease arises from $\sqrt{\Delta_k}$, the width around the saddle point, given in Eq. (A10).) Therefore, the term linear in β dominates the Yukawa term in Eq. (49) when the proper distance is larger than π/H.

In brief, in the adiabatic limit, the two point function contains the usual Yukawa term and a small additional term which is due to pair creation processes. The latter does not decrease like an exponential because, for arbitrary large spatial separation, there is always a pair of onshell quanta which connects the two points since $x_k(t)$ diverges as $k \to 0$. From Eq. (54), we see that the large distance behaviour contains oscillations whose wavelength grows logarithmically. This absence of a distinct pattern is due to the fact that all values of k contribute to the Green function evaluated in the in vacuum.

The lesson of this analysis is that this Green function is not the right object to unravel the space-time correlations. To obtain these correlations, one must introduce a weight in k space so as to keep only a finite range of conformal momenta. This can be done by considering wave-packets. To understand how to introduce wave-packets in second quantization, it is appropriate to pose a physical question whose answer will provide one. Consider the following question: knowing that we start from vacuum and that a particle has been detected at some late time t_0 around x_0, where should one look for its 'partner' ? The appropriate formalism to answer this question consists first, in introducing the projector Π which is associated with the detection of the particle, and second, in computing the value of operators which are conditional to this detection. This was first done in [5] using the formalism developed by Aharonov et al. [19].

B. The projector

When a particle is detected at t_0, x_0 with some momentum $\bar{k}$, the state of the field is 'reduced'. This can be viewed from an axiomatic point of view, or better, à la von Neumann, from the interactions between the field and some additional quantum system which acts as a particle detector. This has been explained with details in [5, 20] and will not be repeated here. In one word, the additional system can be thought to be a bubble chamber. Hence the detection of a particle is the recording of a track in the chamber from which one can deduce both the location and the mean momentum $\bar{k}$ of the particle [21].

For simplicity we consider here only the case of rare pair creation events. Hence we can restrict the analysis to one particle states. We thus assume that a particle is detected around some x_0, at a large future time t_0. Since the detection occurs for $t_0 \gg t_{tp}(\bar{k})$, where

t_{tp} is defined in Eq. (43), one must use the out basis to characterize the local excitation which triggers the external system acting as a particule detector. In $(1+1)$ dimensions, the state of the detected particule is thus specified by

$$|ps\rangle = \int dk f_k a_k^{out\dagger}|0_p out\rangle \,, \tag{55}$$

where the function f_k gives the probability amplitude to find a particle with momentum k. A subscript p (a) added to a ket means that it belongs to the particle (antiparticle) sector in Fock space. Notice indeed that in Eq. (55) nothing has been said about the anti-particle states. It is therefore convenient to introduce the projector associated with the detection of $|ps\rangle$. It is given by

$$\begin{aligned} \Pi_{ps} &:= |ps\rangle\langle ps| \otimes I_a \\ &= \left(\int dk f_k a_k^{out\dagger}|0_p out\rangle \; \langle 0_p out| \int dk' f_{k'}^* a_{k'}^{out} \right) \otimes I_a \,, \end{aligned} \tag{56}$$

where I_a is the identity operator on the antiparticle sector in Fock space.

This projector contains all the information about the detected particle and its partner when the Heisenberg state is the in vacuum. To show this let us apply Π_{ps} on $|0in\rangle$. Using Eq. (18) and developing the exponential to first order, we get

$$\Pi_{ps}|0in\rangle = \frac{1}{\sqrt{Z}}|ps\rangle \otimes \int dk' \; f_{k'}^* \frac{-\beta_{k'}}{\alpha_{k'}} b_{-k'}^\dagger |0_a out\rangle = \frac{1}{\sqrt{Z}}|ps\rangle \otimes |partner\rangle \,. \tag{57}$$

From Eq. (57) we see that the specification of the particle state $|ps\rangle$ univocally fixes the ket $|partner\rangle$, the state of its partner[3]. This is due to the EPR correlations between particle and antiparticle out states which are present in $|0_{in}\rangle$. The projector Π_{ps} gives also the probability to find the chosen particle:

$$P_{ps} = \langle 0in|\Pi_{ps}|0in\rangle = |\langle 0_{in}|0_{out}\rangle|^2 \int dk |f_k|^2 |\frac{\beta_k}{\alpha_k}|^2 \,, \tag{58}$$

which is simply the weighted sum of the probabilities to get a particle with momentum k.

C. The conditional value of the current

The second element of the analysis of [5, 7] is the notion of the expectation value of an operator which is conditional to the detection of the particle described by the ket $|ps\rangle$. Hence instead of considering, as we did in the former Section, the expectation value of $\phi\phi^\dagger$ in the in vacuum, see Eq. (49), we now study its conditional value:

$$\langle \phi(t,\mathbf{x})\phi(t',\mathbf{x}')^\dagger \rangle_{cv} = \frac{\langle 0in|\Pi_{ps}\, \phi(x)\phi(x')^\dagger|0in\rangle}{\langle 0in|\Pi_{ps}|0in\rangle} \,. \tag{59}$$

[3] Eq. (57) is exact, i.e. valid for all values of β/α even though we are presently working in the limit $\beta/\alpha \ll 1$.

The denominator is the probability given in Eq. (58). The numerator is given by

$$\int dk dk' f_k^* f_{k'} (-\frac{\beta_{k'}}{\alpha_{k'}})^* \langle 0out| b^{out}_{-k'} a^{out}_k \phi(x)\phi(x')^\dagger |0in\rangle \,. \tag{60}$$

Using the Bogoliubov transformation $a_k^{in} = \alpha_k a_k^{out} + \beta_k b_{-k}^{out\dagger}$, the commutation relation $[b^{out}_{-k}, b^{in\dagger}_{-k'}] = \alpha_k^* \delta(k-k')$, we get

$$\begin{aligned}\langle 0out| b^{out}_{-k'} a^{out}_k \phi(x)\phi(x')^\dagger |0in\rangle &= \frac{1}{\alpha_k \alpha_{k'}} \phi^{in*}_{-k',a}(x) \phi^{in*}_{k,p}(x') \langle 0out|0in\rangle \\ &\quad - \frac{\beta_k}{\alpha_k} \delta(k-k') \langle 0out|\phi(x)\phi(x')^\dagger|0in\rangle \,. \end{aligned} \tag{61}$$

We have added the subscripts p and a in order to easily identify the wave functions which refer to the particle or the anti-particle. (Notice that in the presence of an electric field, they would be different functions.) Using Eqs.(60,61), the conditional Green function Eq. (59) becomes,

$$\langle \phi(x)\phi(x')^\dagger\rangle_{cv} = G_F(x,x') + \frac{1}{P_{ps}} \left[\int dk \frac{f_k^*}{\alpha_k} \phi^{in*}_{k,p}(x') \right] \left[\int dk' f_{k'} \frac{-\beta^*_{k'}}{\alpha^*_{k'}\alpha_{k'}} \phi^{in*}_{-k',a}(x) \right] \,, \tag{62}$$

where

$$G_F(x,x') = \frac{\langle 0out|\phi(x)\phi(x')^\dagger|0in\rangle}{\langle 0out|0in\rangle} \,, \tag{63}$$

is the in-out propagator properly normalized.

In brief, the conditional value of the Green function splits into a background term which is independent of the chosen particle and a term which is specific to what has been detected.

In the next subsection, we shall use the adiabatic treatment to see how the various semi-classical elements determine the properties of conditional values. Before doing so, we present by several figures the conditional value of the charged density obtained with the exact modes of Section 2. (We could have worked with the conditional value of the energy density ($= \langle \Pi T_{00}(t,x)\rangle / \langle \Pi \rangle$) but since it is more complicated we proceed with the current.) In perfect analogy with Eq. (59), the conditional value of the charge density is given by

$$\langle J(t,x)\rangle_{cv} = \frac{\langle 0in|\Pi_{ps} \hat\phi(t,x) i \overleftrightarrow{\partial_t} \hat\phi(t,x)^\dagger |0in\rangle}{\langle 0in|\Pi_{ps}|0in\rangle} \,. \tag{64}$$

It can be straightforwardly obtained from Eq. (62), and it also splits into two terms

$$\begin{aligned}\langle J(t,x)\rangle_{cv} &= \langle J(t,x)\rangle_{in-out} + \langle J(t,x)\rangle_{ps} \qquad (65) \\ &= \frac{\langle 0out|\hat J(t,x)|0in\rangle}{\langle 0out|0in\rangle} + \frac{1}{P_{ps}} \left[\int dk \frac{f_k^*}{\alpha_k} \phi^{in*}_{k,p}(x) \right] i \overleftrightarrow{\partial_t} \left[\int dk' f_{k'} \frac{\beta^*_{k'}}{\alpha^*_{k'}\alpha_{k'}} \phi^{in*}_{-k',a}(x) \right] \,.\end{aligned}$$

The in-out term will not be analyzed here as it presents no structure. We focus instead on $\langle J(t,x)\rangle_{ps}$ which is specific to the selected state $|ps\rangle$. $\langle J(t,x)\rangle_{ps}$ is complex function.

At first sight, this seems awkward. However, after a little thought one understands that this must be the case. Indeed, the imaginary part of $\langle J \rangle_{ps}$ governs the linear change of the probability P_{ps} to find the chosen particle when one modifies the electric field [6]. (Similarly the imaginary part of the conditional value of $T_{\mu\nu}$ governs the change of P_{ps} when one modifies the background geometry, see Eq. (33) in [7].)

The main features of the conditional value of the current are as follows. First, both the real and imaginary part of $\langle J(t,x) \rangle_{ps}$ vanish for $t \ll t_{tp} = H^{-1} \ln(\bar{k}H/m)$, much before the moment when the $\bar{k}$-particles stop to be relativistic. In other words the $\bar{k}$-configurations are still as in vacuum. Second, $\langle J(t,x) \rangle_{ps}$ gives rise to two well-localized (unit) currents which follow the trajectory of the particle and that of its partner. (The fact that these currents are unity follows from the presence of the denominator in conditional values. The interpretation is clear: *when* a particle has been detected, the *conditional* currents are ± 1.) Third, these two semi-classical currents emerge from wild oscillations which are confined in the creation zone, i.e., in a space-time domain of extension $2/H \times 2/H$ centered around the turning point $t_{tp}(\bar{k})$. In this region the imaginary part of $\langle J(t,x) \rangle_{cv}$ does not vanish but oscillates with the same amplitude as the real part. To explain the origin of these properties it is very useful to return to the adiabatic treatment.

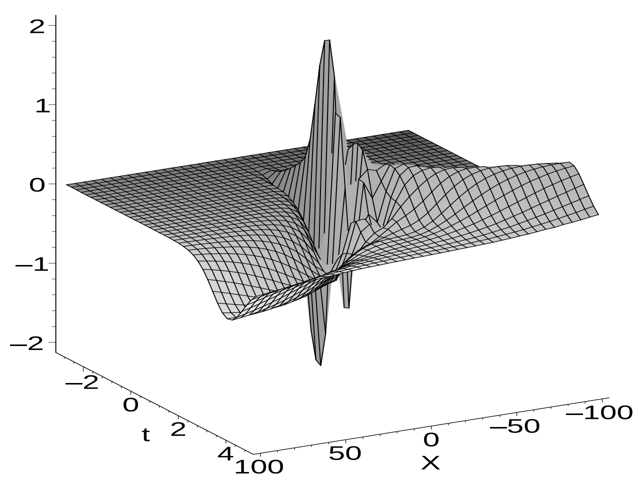

FIG. 4: The real part of the conditional current $\langle J(t,x) \rangle_{ps}$ for $\nu = 3$ as a function of the proper coordinate X and the cosmological time t. The Hubble radius and the Hubble time are equal to one. We have used a nonlinear scale on the z-axis to tame the wild oscillations in the creation zone around the turning point $t_{tp} \simeq -1$. We clearly see that the current vanishes in the past of the creation region and that two semi-classical currents emerge from it. Moreover, they propagate along classical trajectories, see Figure 8.

D. The adiabatic treatment

It is easy to understand why $\langle J(t,x) \rangle_{ps}$ vanishes in the past of the turning point. It follows from the fact that $\langle J \rangle_{ps}$ is the product of two different wave packets, see Eq. (65): one is associated with the incoming particle and the other to its partner. These wave packets are centered along classical trajectories which diverge from each other, see Figures 1 and 7. Hence their product vanishes asymptotically in the past.

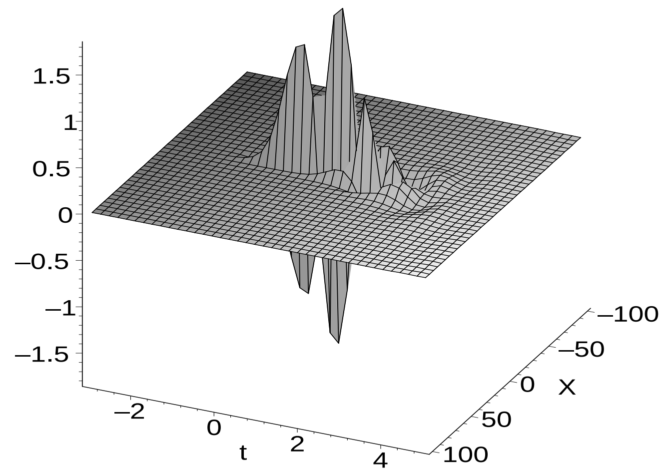

FIG. 5: The imaginary part of the conditional current presented in the former Figure. We have again used a nonlinear scale on the z-axis to tame the fluctuations which are of the same order as the real ones. The imaginary part takes significant values only in the creation region. This means that once the particles are on-shell, the conditional current $\langle J(t,x)\rangle_{ps}$ behaves as a classical current.

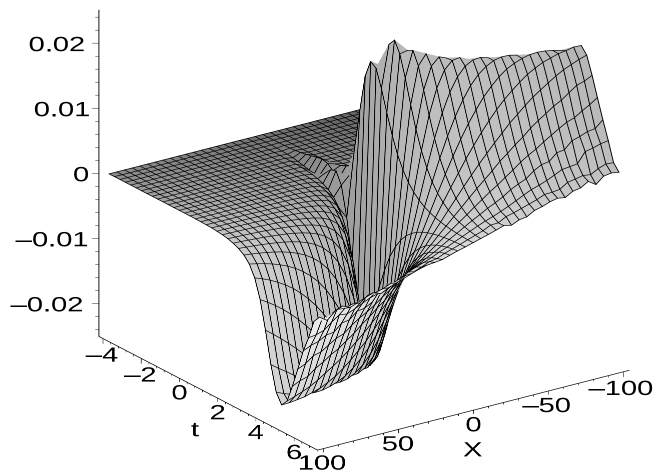

FIG. 6: The real value of the conditional current far away from the adiabatic regime, for $\nu = 0.01$, as a function of X and t. Even though the wave packets are much broader (this is hardly surprising since the mass is now much smaller that the Hubble radius $= 1$) one sees that the properties of the conditional value are robust. It still vanishes in the past of the turning point and still gives rise to two semi-classical currents following classical trajectories.

To understand the properties in the future of the turning point, namely why it describes two localized and real currents, it is appropriate to work with out modes since they are

WKB in that period. Using Eq. (12), the numerator of $\langle J(t,x)\rangle_{ps}$ is

$$\left[\int dk \frac{f_k^*}{\alpha_k}\phi_{k,p}^{in*}(x)\right] i\overleftrightarrow{\partial_t}\left[\int dk f_k \frac{\beta_k^*}{|\alpha_k|^2}\phi_{-k,a}^{in*}(x)\right] =$$

$$\left[\int dk f_k \phi_{k,p}^{out}(x)\right]^* i\overleftrightarrow{\partial_t}\left[\int dk f_k |\frac{\beta_k}{\alpha_k}|^2 \phi_{k,p}^{out}(x)\right] \tag{66a}$$

$$+\left[\int dk f_k^* \frac{\beta_k}{\alpha_k}\phi_{-k,a}^{out}(x)\right] i\overleftrightarrow{\partial_t}\left[\int dk f_k^* \frac{\beta_k}{\alpha_k}\phi_{-k,a}^{out}(x)\right]^* \tag{66b}$$

$$-\left[\int dk f_k^* \phi_{k,p}^{out*}(x)\right] i\overleftrightarrow{\partial_t}\left[\int dk f_k \frac{\beta_k^*}{\alpha_k^*}\phi_{-k,a}^{out*}(x)\right] \tag{66c}$$

$$-\left[\int dk f_k^* \frac{\beta_k}{\alpha_k}\phi_{-k,a}^{out}(x)\right] i\overleftrightarrow{\partial_t}\left[\int dk f_k |\frac{\beta_k}{\alpha_k}|^2 \phi_{k,p}^{out}(x)\right] . \tag{66d}$$

In Eq. (66a), in the first bracket one finds the wave function of the particle,

$$F_p(t,x;t_0,x_0,\bar{k}) = \langle 0_p out|\hat{\phi}(t,x)|ps\rangle = \int dk\ f_k\ e^{ikx}\phi_k^{T_{out}}(t)\,. \tag{67}$$

To evaluate it, we use for simplicity a gaussian wave packet

$$f_k = e^{-\frac{(k-\bar{k})^2}{4\sigma^2}} e^{-ikx_0} e^{-i\int_{t_0}^{T_{out}} dt\ \omega_k(t)}\,. \tag{68}$$

The phases are chosen so that the wave-function is maximum at (t_0, x_0). We proceed again with a saddle-point analysis. Up to an irrelevant phase, we get

$$F_p(x,t) \simeq \sqrt{\frac{2\pi}{\partial_k^2\Phi(k^*)}}e^{\Phi(k^*)}\,, \tag{69}$$

where

$$\Phi(k) = -\frac{(k-\bar{k})^2}{4\sigma^2} + ik(x-x_0) - i\int_{t_0}^{t} dt'\ \omega_k(t')\,. \tag{70}$$

The saddle-point k^* is the solution of

$$\frac{d\Phi}{dk}|_{k^*} = 0 = -\frac{k^*-\bar{k}}{2\sigma^2} + i(x-x_0) - i\int_{t_0}^{t} dt'\ \partial_k\omega_k(t')|_{k^*}\ . \tag{71}$$

If one neglects the spread, i.e. $\sigma \to \infty, k = \bar{k}$, one recovers the classical equation of motion Eq. (A7). The role of the spread is to damped the wave packet when k^* differs from the mean momentum $\bar{k}$. We thus look for a solution of the type $k^* = \bar{k} + \delta k$. Expanding Eq. (71) around $\bar{k}$ to lowest order in δk, we get

$$\delta k = i2\Sigma_{\bar{k}}^2\left(x - x_0 - \Delta x_{\bar{k}}(t_0,t)\right), \tag{72}$$

where $\Delta x_{\bar{k}}^2(t_0,t)$ is the classical displacement from t_0 to t when $k = \bar{k}$.

$$\frac{1}{2\Sigma_{\bar{k}}^2(t)} = -\partial_k^2\Phi_k|_{\bar{k}} = \frac{1}{2\sigma^2} + i\partial_k\Delta x_k(t_0,t)|_{k=\bar{k}}\,, \tag{73}$$

gives the time dependent spread of the particle's wave packet. When expanding $\Phi(k^*)$ around $\bar{k}$ one gets

$$F_p(x,t) \sim \sqrt{\frac{2\pi}{\partial_k^2\Phi(\bar{k})}} e^{\Phi(\bar{k})} \exp(-2\Sigma_{\bar{k}}^2[x - x_0 - \Delta x_{\bar{k}}(t_0,t)]^2) \; . \qquad (74)$$

As expected, the wave-packet is a gaussian centered along the classical trajectory characterized by the mean momentum $\bar{k}$.

The second bracket in Eq. (66a) is $F_p(x,t)$ multiplied by a k-independent real factor, $|\frac{\beta_k}{\alpha_k}|^2$. Hence the first line of the r.h.s in Eq. (66) gives the current carried by the wave packet described by the asymptotic state $|ps\rangle$. This explains why asymptotically $\langle J\rangle_{ps}$ is real and concentrated along the trajectory associated with the chosen particle state. We shall see below that the same reasoning will apply to the partner's wave function. The role of the normalization factor $|\frac{\beta_k}{\alpha_k}|^2$ is to guarantee that the current carried by $\langle J\rangle_{cv}$ is one, thanks for the presence of the denominator P_{ps} in Eq. (62).

In the third and the fourth lines of Eq. (66), one finds products of the particle and the antiparticle wave-packets. For late times they vanish in the adiabatic limit because the classical trajectories do not overlap. Their role is to guarantee that $\langle J\rangle_{ps}$ vanishes in the past, and to contribute to the complex oscillations of large amplitude ($= |\alpha/\beta|$) in the creation zone. To show that there is no significant overlap in these products, we now compute the properties of the wave-packet of the partner which appears in the second line in the r.h.s. of Eq. (66).

In Eq. (66b), we find the wave function of the partner and its complex conjugate. Hence the current they describe is real. (In fact, this follows from causality: when t, x is separated from the support of Π by a space-like distance, Π and $\hat{J}$ commute, hence $\langle J\rangle_{ps}$ is real.) Having understood the asymptotic reality properties of the conditional current $\langle J\rangle_{ps}$, it is now interesting to determine the trajectory followed the partner and what are the parameters which determine it. As for the particle, it is appropriate to use a saddle-point approximation. Explicitly, one has:

$$F_a(t,x) = \langle 0_a out|\hat{\phi}(t,x)^\dagger|partner\rangle = \int dk\; f_k^* \frac{-\beta_k}{\alpha_k} e^{-ikx} \phi_k^{T_{out}}(t) = \int dk\; e^{\Psi(k)} \, , \qquad (75)$$

where

$$\begin{aligned} \Psi(k;x,t,x_0,t_0) &= -\frac{(k-\bar{k})^2}{4\sigma^2} - ik(x-x_0) + i2\psi_k' + i\int_t^{t_0} dt'\; \omega_k(t') \, , \qquad (76) \\ &= -\frac{(k-\bar{k})^2}{4\sigma^2} - ik(x-x_0) - i\left(\int_{t_c(k)}^{t_0} dt'\; \omega_k(t') + \int_{t_c(k)}^{t} dt'\; \omega_k(t')\right) . \end{aligned}$$

In the first line we have used the complex phase ψ_k' introduced in Eq. (40) and we have identified T_{out} with the late detection time t_0. The unusual sum of actions we obtain from ψ_k' has a clear meaning. It gives the phase accumulated from the detection of the particle at time t_0 to the location of the partner at time t when going *backwards* in time so as to pass through the creation process which occurred at $t_c(k)$.

The behaviour of F_a is governed by the saddle-point k^* defined by $\frac{d\Psi}{dk}|_{k^*} = 0$. As in the previous analysis, when one searches for the characterization of the wave function close

to its maximum, all quantities can be evaluated for $k = \bar{k}$. The condition $\frac{d\Psi}{dk}|_{\bar{k}} = 0$ thus gives the trajectory of the partner:

$$x_a(t) = x_0 + \partial_k \left[2\psi'_k + \int_t^{t_0} dt' \; \omega_k(t') \right] |_{k=\bar{k}} \, . \tag{77}$$

From this equation, the classical equation of motion of the particle, Eq. (A6) evaluated for $k^* = \bar{k}$, we get

$$\begin{aligned} x_a(t) - x_p(t) &= -2\partial_k \int_{t_c(k)}^{t} dt \; \omega_k(t)|_{\bar{k}} \, , \\ &= -2x_k(t)|_{k=\bar{k}} \, . \end{aligned} \tag{78}$$

This is the central equation of this Section. It tells us that, at time t, the (mean) conformal separation between the $\bar{k}$-particle and its partner *only* depends on the HJ action evaluated from the creation time $t_c(\bar{k})$ to t. None of the details of the wave packet enters in it. This result agrees we what we found in Eq. (53). Notice that it would have been also obtained if we used the conditional value of energy density rather than $\langle J(t,x)\rangle_{ps}$. Moreover, when working outside the semi-classical regime, one still reaches the same conclusion, see Figure 6.

Expanding the solution around $\bar{k}$, one gets

$$F_a(x,t) \simeq \sqrt{\frac{2\pi}{\partial_k^2 \Psi(\bar{k})}} e^{\Psi(\bar{k})} \exp\left(-2\Sigma_{\bar{k},a}^2 (x - x_a(t))^2\right) \, . \tag{79}$$

As in Eq. (74), we obtained the semi-classical wave function evaluated at $k = \bar{k}$ times gaussian centered along the classical trajectory of the partner. They are however some differences. First $e^{\Psi(\bar{k})}$ contains a factor $e^{-\pi m/H} = \beta/\alpha$. Hence the current carried by the second line in the r.h.s of Eq. (62) is equal to -1 when taking into account the denominator P_{ps}. It is thus equal and opposite to that carried by the particle wave function in the first line. Second, the interesting novelty concerns the spread. It is given by

$$\frac{1}{2\Sigma_{\bar{k},a}^2(t)} = \frac{1}{2\sigma^2} - i2\partial_k x_a(t)|_{k=\bar{k}} = \frac{1}{2\sigma^2} - i\left[\Delta_k(t) + \Delta_k(t_0)\right]|_{k=\bar{k}} \, , \tag{80}$$

where $\Delta_k(t)$ is given in Eq. (A10). As in Eq. (73), it is the 'susceptibility' of the HJ trajectory which determines the time dependence of the spread. The novelty is that this spread now depends on the second derivative of ψ'_k. Hence it is intrinsic to the pair creation process. From it we can derive the notion of minimal wave-packets. Consider $\delta_{\bar{k},a}(t_0)$, the spread in x of $F_a(x,t)$ at time $t_0 \gg t_{tp}(\bar{k})$. It is given by

$$\frac{1}{\delta_{\bar{k},a}^2(t_0)} = \Re\left\{\Sigma_{\bar{k},a}^2(t_0)\right\} = \frac{1/\sigma^2}{1/\sigma^4 + 4\Delta_{\bar{k}}^2(t_0)} \, . \tag{81}$$

Its minimum value is obtained for $\sigma^{-2} = 2\Delta_{\bar{k}}(t_0) \simeq 2m/H\bar{k}^2$, see Eq. (A11). Then $k^2\delta_{k,a}^2 = 8m/H$ for all k and for late time. We thus see that the otherwise arbitrary spread in k $(= \sigma)$ is now fixed by the asymptotic value of the second derivative of the HJ action evaluated from the creation time $t_c(k)$. In this respect, minimal wave packet are

intrinsic to the creation process. Moreover, since they cover a space-time zone of proper extension $\Delta t \times \Delta x_P = (mH)^{-1}$, the number of non overlapping minimal wave packets in a space time box correctly gives the number of 'relevant' modes which govern the vacuum instability in Eq. (47). Hence they provide a natural basis for describing pair creation processes[4].

This result also justifies the fact that we neglected the product of wave functions of the particle and its partner in Eqs. (66c, 66d). Indeed, the asymptotic separation of the classical trajectories of the particle and its partner is $x_a(t_0) - x_p(t_0) \sim 2m/Hk$. Thus one has

$$\frac{x_a(t_0) - x_p(t_0)}{\delta_{k,a}} \sim \sqrt{\frac{m}{H}}\,. \tag{82}$$

Therefore, for minimal wave packets and in the adiabatic regime $m/H \gg 1$, the asymptotic value of the overlaps in Eqs. (66c, 66d) are exponentially small.

Finally we characterize the size of the space-time region in which the conditional current $\langle J \rangle_{ps}$ does *not* behaves semi-classically. Its location in time is centered near the turning point time $Ht_{tp} = \ln(kH/m)$ and has a duration of about 4 Hubble times. For larger time lapses, the non-adiabaticity of the propagation, governed by $\lambda_k(t)$, decreases exponentially fast. Hence non-adiabatic effects are concentrated in that lapse. The spread in space of this region is then given by the distance between the two partners at t_{tp}:

$$\Delta x_{k,creation} \doteq 2 x_k(t)|_{t=t_{tp}(k)} = 2\,\frac{\sqrt{2}}{Ha(t_{tp})}\,. \tag{83}$$

Hence the corresponding proper distance is given by $\Delta x_P = 2\sqrt{2}/H$. Thus, at the creation time and for all values of k, the partner is outside the Hubble radius centered around the particle by a factor of $2\sqrt{2}$, see Figure 9. As long as the inflation goes on, the two members in each pair stay outside the Hubble radius.

Hence, from all measurements performed within a Hubble patch, one would conclude that the created particles are described by an incoherent density matrix, as in the case of Hawking radiation when performing measurements far away from the black hole [6, 8]. However, the difference with black hole physics is that the Hubble radius grows when inflation stops. Hence progressively the particles will join their partner within the same Hubble radius. Then, the coherence of the in vacuum is recovered within a patch and interference patterns can be obtained.

IV. LINK WITH PHYSICAL COSMOLOGY

When applying the preceding analysis of conditional values to primordial gravitational waves or primordial density fluctuations, one faces new features. In this Section, we briefly present them. A more detailed analysis will be presented in a forthcoming work.

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[4] This is again very similar to what is found when studying the Schwinger effect [6]. In fact the whole analysis we just performed can be applied as such to study that case. One should simply work in the homogeneous gauge $A_t = 0, A_x = Et$, and replace our frequency $\omega_k^2 = m^2 + k^2/a(t)^2$ by the relevant one: $\omega_k^2 = m^2 + (k - Et)^2$.

The first novelty arises from the fact that primordial waves are described by massless minimally coupled fields. Hence, their frequency Ω_k becomes imaginary when the modes exit the Hubble radius. Thus out modes no longer exist. This can be seen from Eq. (22) and the fact that ν is now imaginary. Instead in modes are still perfectly defined as in Eq. (19) because in the past the physical momentum still becomes much smaller than the Hubble radius. To obtain well-defined out modes (which are necessary to specify the state at late times) one should stop inflation and consider the following adiabatic era so as to let the modes re-enter the Hubble radius and start to re-oscillate. This does not raise any new conceptual difficulty but the necessity of dealing with two different periods of expansion prevents to obtain a simple and analytical description of modes throughout their evolution (unless one neglects, as usually done, the decaying mode.)

The second novelty concerns the occupation number ($= |\beta_k|^2$). It becomes extremely large during the period wherein the modes do not oscillate. Thus one inevitably works in the limit $\beta_k/\alpha_k \to 1$. In this limit, Eq. (50) becomes

$$\langle 0_{in}|\hat{\phi}(t,\mathbf{x})\hat{\phi}(t,\mathbf{0})^\dagger|0_{in}\rangle = 2|A|^2 \int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\mathbf{kx}}}{k^4}\,|\phi_k^{out}(t)|^2\left(1-\cos 2\psi_k(t)\right)\,, \tag{84}$$

since in inflation one has $|\alpha_k|^2 = |A|^2/k^4$ when the slow-roll condition $\partial_t H/H^2 \ll 1$ is satisfied [2]. We have introduced the time dependent phase $\psi_k(t)$ given by

$$e^{i\psi_k(t)} = e^{i\psi_k}\,\frac{\phi_k^{out}(t)}{|\phi_k^{out}(t)|}\,. \tag{85}$$

As in the adiabatic case $\beta/\alpha \ll 1$, see Eq. (50), $\psi_k(t)$ governs the spatial properties of the Green function. [Notice that this phase also governs the spherical harmonic coefficients $C_l(t_r,t_e)$ which characterize the two-point function at emission time t_e as seen at reception time t_r. They are given by [2]

$$\begin{aligned} C_l(t_r,t_e) &= \frac{(4\pi)^2}{9}\int_0^\infty dk\,k^2 j_l^2(k\Delta\eta)\;\frac{|\phi_k^{in}(t_e)|^2}{a(t_e)^3} \\ &= \frac{|A|^2}{a(t_e)^2}\,\frac{(4\pi)^2}{9}\int_0^\infty \frac{dk}{k}\,j_l^2(k\Delta\eta)\;\frac{1-\cos 2\psi_k(t_e)}{k^2}\,, \end{aligned} \tag{86}$$

where $\Delta\eta = \eta(t_r)-\eta(t_e)$ is the lapse of conformal time and where j_l is the spherical Bessel function. We have also used the fact that in a radiation dominated universe, the rescaled field ϕ introduced after Eq. (2) satisfies $k|\phi_k^{out}(t_e)|^2/a(t_e) = 1/2$ when $m=0$. When the function which multiplies j_l^2 in the integrand is k-independent, $l(l+1)C_l$ is independent of l. Hence, the dependence in l of $l(l+1)C_l$ is governed by $\psi_k(t_e)$.]

The difficulty related to the high occupation number only concerns the use of conditional values. Indeed, it is now meaningless to specify the occupation number as we did it in Eq. (56). One should now work with a new projector which specifies that a classical wave has been detected. This projector should thus be built on the field amplitude itself.

Instead of Eq. (56), it could be now of the form[5]

$$\tilde{\Pi} = \int dk'\, f_{k'} \hat{a}^{\dagger out}_{k'} \int dk\, f^*_k \hat{a}^{out}_k \,. \tag{87}$$

Perhaps the simplest way to understand its meaning is to consider the interaction of the field $\hat{\phi}(t,x)$ with an additional system which acts as a wave detector, and to work to second order in the coupling by letting the dynamics proceed. We refer to Section 3.A of [20] for an explicit example.

Using the new projector, in the place of Eq. (62), one obtains

$$\begin{aligned} \frac{\langle 0in|\tilde{\Pi}\, \hat{\phi}(x)\hat{\phi}(x')^\dagger|0in\rangle}{\langle 0in|\tilde{\Pi}|0in\rangle} &= \langle 0_{in}|\hat{\phi}(x)\hat{\phi}(x')^\dagger|0_{in}\rangle \\ &+ \frac{1}{\tilde{P}}\left[\int dk' f_{k'}(-\beta^*_{k'})\phi^{*in}_{-k',a}(x)\right]\left[\int dk f^*_k \alpha^*_k \phi^{*in}_{k,p}(x')\right] , \end{aligned} \tag{88}$$

where $\tilde{P}$ is the probability to detect the chosen wave. It is given by $\tilde{P} = \langle 0in|\tilde{\Pi}|0in\rangle$. As in Eq. (62), the conditional value splits into a background term and a term specific to the wave which has been detected. We note two differences. The background term is now the Green function evaluated in the in vacuum. This results from the fact that the new projector specifies much less the final configurations that the first one did. Indeed the latter stipulated that only one particle was found, hence the appearance of the out vacuum in Eq. (62). The other novelty concerns the second term. Each integrand has been multiplied by $|\alpha_k|^2 = n_k + 1$, which results from the Bose statistics of the field. The crucial point is that the presence of this new factor contains no phase. Hence the locus of constructive interferences is unchanged with respect to we had in Eq. (62). Thus the results found in Section 3.4 still apply, no matter how large is the occupation number. In fact the analysis would also apply to configurations which are described by a classical probability distribution. In this case however, one would obtain conditional values which are real. This should cause no surprise since the correspondence between quantum two-point functions and classical correlators is obtained when using anti-commutators in quantum settings. In our case, the 'classical' conditional two-point function would be given by Eq. (88) when replacing $\tilde{\Pi}\,\phi\phi^\dagger$ by the anti-commutator $\{\tilde{\Pi}, \phi\phi^\dagger\}_+$. This leads to a real conditional value which coincides to that one would obtain using classical settings.

In brief, the usefulness of considering conditional values, and not only expectation values, still applies in the classical limit. As in the adiabatic regime, conditional values give us the spatial distribution of correlations relative to a restricted set of configurations, those which have been selected by the projector. It is this filtering procedure which brings much more detailed information as it eliminates the washing out mechanism present in expectation values. One can thus envisage to compute the conditional values of the harmonic coefficients as they would contain more information as well. They would now depend on m, the projection of l on a given axe. They are given by the usual definition

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[5] It should be clear to the reader that this is not the only possibility. Since we are working in the classical limit, one could equally work with an anti-commutator. When the differences between two options are given in terms of commutators, they are irrelevant in the classical limit since they are order 1 whereas anti-commutators are order of $|\beta|^2 \gg 1$.

when replacing the expectation value of $\phi(t_e, \mathbf{x})\phi(t_e, \mathbf{x}')$ by its conditional value. The relevance of applying these concepts to observational data is presently under consideration.

V. CONCLUSIONS

In this article, we have analyzed the space-time distribution of the correlations which are induced by pair creation processes in cosmology. Our analysis applies both to the adiabatic regime, where the creation rate is exponentially suppressed, and to the opposite regime where the mean occupation number can be arbitrary large. The reason is that the space-time distribution is determined by products of the form $\phi^2_{k,out}\beta_k/\alpha_k$, see Eq. (49) and Eq. (84). Irrespectively of the occupation number ($= |\beta^2_k|$) it is thus the k-dependence of the phase of β_k/α_k which matters.

However the space-time distribution exhibited in expectation values shows no specific properties because all momenta contribute with equal weight in the vacuum. Therefore, to isolate the correlations associated with a given pair of particles, we studied the value of the current which is conditional to the detection of one member of that pair, see Eq. (65). We found that each pair gives rise to two semi-classical currents which are localized along the trajectories of the particle and its partner. These currents emerge from wild oscillations in a creation zone of space-time extension $\sim 4/H \times 4/H$ which is centered around the time at which the particles cease to be relativistic. We then showed that the properties of these conditional values are intrinsic to the creation process in that they follow from the Hamilton-Jacobi action evaluated from the creation time $t_c(k)$. This action and its derivatives with respect to k govern indeed the statistical weight of amplitudes, Eq. (39), the space-time separation between the particle and its partner, Eq. (78), and the spread of their wave functions, Eq. (80).

Even though this semi-classical analysis is a priori restricted to the adiabatic regime, the results it explains are also found very far from it, as shown in Figure 6. In particular each pair of modes still gives rise to a pair of localized energy density fluctuations whose properties are fixed by the creation process. Thus the space-time information encoded in conditional values is still richer than that of expectation values where some details are washed out by the averaging procedure. The robustness of the space time properties far from the adiabatic regime is an interesting fact which deserves further study. To clarify this point, to analyze what happens when dealing with massless minimally coupled fields (i.e., the relevant case for describing primordial gravitational waves), and to determine the possible use of conditional values in analyzing primordial spectra are three motivations for further work.

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APPENDIX A: THE HAMILTON-JACOBI ACTION

In this Appendix, we compute the various integrals obtained when working in the adiabatic approximation. They are all expressed in terms of the Hamilton-Jacobi action and its derivatives. For simplicity we work in a flat de Sitter space in $1+1$ dimensions.

In the RW metric Eq. (1), the Hamilton-Jacobi equation reads [22]

$$-(\partial_t S)^2 + \frac{1}{a^2}(\partial_{\mathbf{x}} S)^2 + m^2 = 0 \tag{A1}$$

where $\mathbf{k} = \partial_{\mathbf{x}}\mathbf{S}$ is the comoving momentum. Since it is conserved, the action separates as

$$S(\mathbf{x}, t; \mathbf{x_0}, t_0) = \mathbf{k}\cdot(\mathbf{x} - \mathbf{x_0}) - S_k(t, t_0)\,, \tag{A2}$$

where

$$(\partial_t S_k)^2 = \omega_k(t)^2 = \frac{k^2}{a^2} + m^2\,. \tag{A3}$$

The solution with positive energy which vanishes at t_0 can be written as

$$S_k(t, t_0) = \int_{t_0}^{t} dt'\,\omega_k(t') = F_k(t) - F_k(t_0)\,, \tag{A4}$$

where the primitive is given by

$$F_k(t) = -\frac{m}{H}\left[\frac{1}{u}\sqrt{1+u^2} - arcsinh(u)\right]\,, \tag{A5}$$

and where $u(t) = ma(t)/k$.

The classical trajectory keeping the end-points fixed is given by

$$\partial_{\mathbf{k}} S(\mathbf{x}, t; \mathbf{x_0}, t_0) = 0\,. \tag{A6}$$

It gives $\mathbf{x}(t) = \mathbf{x_0} + \Delta\mathbf{x}_{cl}(t, t_0)$. The classical displacement from t_0 to t is

$$\Delta\mathbf{x}_{cl}(t, t_0) = \int_{t_0}^{t} dt'\;\partial_{\mathbf{k}}\omega_k(t') = \left[-\frac{sgn(\mathbf{k})}{Ha(t)}\sqrt{1+\frac{m^2 a(t)^2}{k^2}}\right]_{t_0}^{t} = x_k(t) - x_k(t_0)\,. \tag{A7}$$

In preparation for the semi-classical description of pair production, we have introduced

$$x_k(t) = \partial_{\mathbf{k}} F_k(t) = -\frac{sgn(\mathbf{k})}{Ha(t)}\sqrt{1+\frac{m^2 a(t)^2}{k^2}}\,. \tag{A8}$$

As we shall see, it will determine the position of the particle in a pair with respect to the 'center of mass' of that pair, called $\bar{x}$ in Section 2.2. In other words $2x_k(t)$ is the conformal distance between the two members in *all* pairs of conformal momenta k, see Section 3.4 and Figures 7 and 8. Its asymptotic behaviour is

$$\text{for}\quad t\to-\infty,\quad x_k \sim -\frac{sgn(\mathbf{k})}{Ha(t)} \to -sgn(\mathbf{k})\infty \tag{A9a}$$

$$\text{for}\quad t\to+\infty,\quad x_k \sim -sgn(\mathbf{k})\frac{m}{H}\frac{1}{k} \tag{A9b}$$

Finally, wave-packets tend to spread because of the non-linear dependence of S_k with respect to k. In the semi-classical approximation, the growth of the spread is governed by the susceptibility of the classical trajectory with respect to $\mathbf{k}$:

$$\Delta_k(t) = \partial_{\mathbf{k}} x_k = \frac{m^2}{Hk^2}\frac{1}{(k^2/a^2+m^2)^{1/2}}\,. \tag{A10}$$

Asymptotically, one finds

$$\begin{aligned} &\text{for } t \to -\infty\ , \quad \Delta_k \to 0\,, \\ &\text{for } t \to +\infty\ , \quad \Delta_k \to \frac{m}{H}\frac{1}{k^2}\,. \end{aligned} \tag{A11}$$

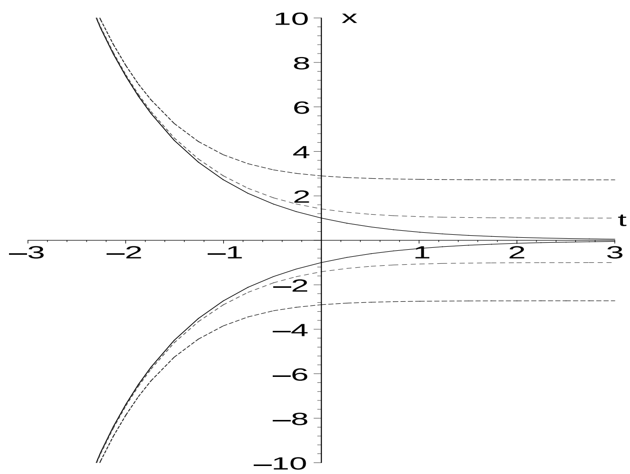

FIG. 7: The continuous line corresponds to the Hubble radius centered around $x = 0$. x is the conformal distance and t the cosmological time. In dotted and dashed lines, we have drawn two pairs of classical trajectories with respective comoving momenta $k = \pm 1$ and $k = \pm 1/e$. In each pair, the two trajectories have been placed symmetrically with respect to $x = 0$ and the arbitrary distance between them has been fixed by the result delivered by the pair creation process, that is $2x_k(t)$ of Eq. (A8).

APPENDIX B: SQUEEZING FORMALISM

In this appendix, we make contact between the Bogoliubov formalism and the squeezing formalism.

The in and out annihilation operators and vacua are respectively related by

$$a_{\mathbf{k}}^{in} = \mathcal{S} a_{\mathbf{k}}^{out} \mathcal{S}^\dagger\ , \tag{B1a}$$

$$|0, in\rangle = \mathcal{S}|0out\rangle\ , \tag{B1b}$$

where $\mathcal{S}$ is the scattering-matrix is defined by

$$\mathcal{S} = U(T_{in}, T_{out})\ . \tag{B2}$$

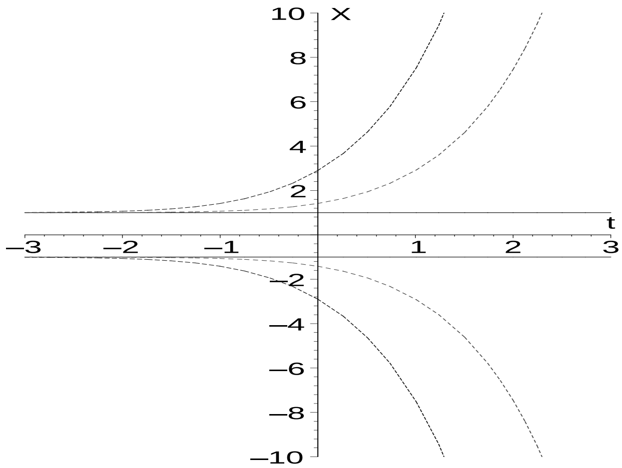

FIG. 8: The same trajectories as in the former figure but now represented in terms of the proper distance $X = a(t)x$ measured from $x = 0$. In our units, the Hubble radius is equal to one.

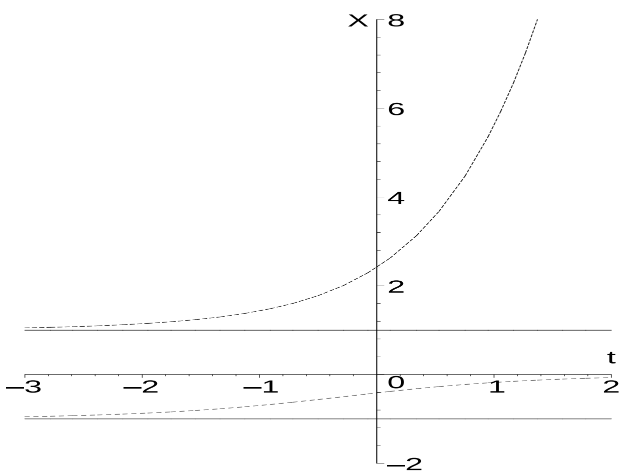

FIG. 9: The trajectory corresponding to $k = 1$ now represented in terms of the proper distance from the asymptotic position of the particle with positive momentum. The Hubble radius is still equal to one. In the remote past, the partner's trajectory hugs the outer side of the horizon, as in black hole radiation [6].

$U(t)$ is the evolution operator, the time ordered exponential of $-i\int_{T_{in}}^{t} dt\ H(t)$.

In the squeezing formalism, the evolution operator is decomposed into the product of a rotation operator R and a squeezing operator S, see [23]. Since the components of the field decouple, one can thus work at fixed $\mathbf{k}$ and write

$$\mathcal{S}_{\mathbf{k}} = R_{\mathbf{k}} S_{\mathbf{k}} \,. \tag{B3}$$

When expressing these two operators in terms of out creation and destruction operators, one has

$$R_{\mathbf{k}}(\theta) = \exp\left\{-i\theta\left(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + b_{-\mathbf{k}}^{\dagger}b_{-\mathbf{k}}\right)\right\} \,, \tag{B4a}$$

$$S_{\mathbf{k}}(r,\phi) = \exp\left\{r\left(e^{-i2\phi}b_{-\mathbf{k}}a_{\mathbf{k}} - cc\right)\right\} \,. \tag{B4b}$$

Their action on annihilation and creation (out) operators are:

$$R(\theta)a_{\mathbf{k}}R^\dagger(\theta) = e^{i\theta}a_{\mathbf{k}}, \quad R(\theta)b_{-\mathbf{k}}R^\dagger(\theta) = e^{i\theta}b_{-\mathbf{k}} \tag{B5}$$

and

$$S(r,\phi)a_{\mathbf{k}}S^\dagger(r,\phi) = \mathrm{ch}(r)a_{\mathbf{k}} + e^{i2\phi}\mathrm{sh}(r)b^\dagger_{-\mathbf{k}} \tag{B6}$$

We now give the expression of the in vacuum in term of out states. Starting from

$$|0_{\mathbf{k}}, in\rangle = R_{\mathbf{k}}(\theta_k)S_{\mathbf{k}}(r_k,\phi_k)\ |0_{\mathbf{k}}, out\rangle\ , \tag{B7}$$

and using the general relation

$$R^\dagger(\theta)S(r,\phi)R(\theta) = S(r,\phi+\theta)\ , \tag{B8}$$

one obtains

$$|0_{\mathbf{k}}, in\rangle = S_{\mathbf{k}}(r_k,\phi_k-\theta_k)\ |0_{\mathbf{k}}, out\rangle\ . \tag{B9}$$

One can further decompose the squeezing operator into the product of three operators:

$$\begin{aligned} S_{\mathbf{k}}(r_k,\phi_k) \ &= \ \frac{1}{\mathrm{ch}r_k}\exp\{-e^{+i2\phi_k}\mathrm{th}r_k\ a^\dagger_{\mathbf{k}}b^\dagger_{-\mathbf{k}}\} \\ &\times\exp\left(-2\ln(\mathrm{ch}r_k)\left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + b^\dagger_{-\mathbf{k}}b_{-\mathbf{k}}\right)\right)\ \exp\left(e^{-i2\phi_k}a_{\mathbf{k}}b_{-\mathbf{k}}\right)\ . \end{aligned} \tag{B10}$$

Regrouping Eq. (B9) and Eq. (B10), one finally gets[6]

$$|0_{\mathbf{k}}, in\rangle = \frac{1}{\mathrm{ch}r_k}\exp\left(-e^{+i2(\phi_k-\theta_k)}\mathrm{th}r_k\ a^\dagger_{\mathbf{k}}b^\dagger_{-\mathbf{k}}\right)\ |0_{\mathbf{k}}, out\rangle \tag{B11}$$

To make contact with Eq. (18), we first identify the Bogoliubov coefficients defined by

$$a^{in}_{\mathbf{k}} = \mathcal{S}a^{out}_{\mathbf{k}}\mathcal{S}^\dagger = \alpha_k a^{out}_{\mathbf{k}} + \beta_k b^{out\dagger}_{-\mathbf{k}}\ , \tag{B12}$$

We use the relations Eq. (B5) and Eq. (B6) and get

$$\alpha_k = e^{i\theta_k}\mathrm{ch}(r_k)\ , \tag{B13a}$$
$$\beta_k = e^{-i\theta_k}e^{i2\phi_k}\mathrm{sh}(r_k)\ . \tag{B13b}$$

Then one can check that inserting Eq. (B13) in Eq. (18) one recovers Eq. (B11).

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[6] Notice that we have normal ordered the products of operators in Eq. (B4). Had we not done it, Eq. (B11) would have been modified by a phase ($= e^{-i\theta_k}$) which corresponds to the zero-point energy of the two modes states.

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